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Drainage From a Uniform Soil Layer on a Hillslope

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A simple hillslope hydrological model predicting discharge from sloping shallow soils is analyzed. Unlike most existing hillslope models, this model is fully analytic and thus straightforward to apply. It compares favorably to more complex models, and its application is illustrated for experimental data collected at the Coweeta Hydrological Laboratory.

INTRODUCTION

Mechanisms of streamflow generation have received considerable attention in hydrological discussion for many years. Algorithms for predicting stream flow have, in the main, relied on *Horton*'s [1933] explanation of runoff generation in spite of persistent failure to observe this phenomenon in a wide variety of natural and forested environments (see, for instance, *Hewlett and Hibbert* [1965]; *Dunne* [1983]; and *Ward* [1984] for excellent reviews). It is now recognized that Horton's theory represents only one extreme in a spectrum of runoff mechanisms involved in generating stream flow [*Dunne*, 1983; *Smith and Hebbert*, 1983]. The other extreme is subsurface flow.

Recently, *Sloan and Moore* [1984] published results comparing the drainage and cumulative drainage profiles predicted by five subsurface flow models. The models in their study included *Nieber and Walter* [1981] and *Nieber*'s [1982] one- and two-dimensional finite element solution of Richards' equation, the kinematic wave model [*Beven*, 1981], and two storage-discharge (kinematic storage and Boussinesq storage) models developed by Sloan and Moore. All the models are basically numerical, requiring evaluation on a time step basis. The application of each model was compared to experimental observations collected from a small and well-defined hillslope segment at the Coweeta Hydrological Laboratory in the southern Appalachian mountains [*Hewlett and Hibbert*, 1963]. A concrete-lined soil trough, 0.92 × 0.92 × 13.72 m, was constructed on a 40% slope and filled with recompacted C horizon (Halewood) forest soil. The soil was soaked using sprinklers, covered with plastic to prevent evaporation, and allowed to drain. The Coweeta study is of practical interest because it provides data that can be used to evaluate the ability of subsurface models to simulate flow in shallow soils overlying a steeply sloping impermeable bed [*Sloan and Moore*, 1984].

The purpose of this note is to present an alternate hillslope hydrologic model which, unlike the others, is fully analytic and can be used to determine the drainage and cumulative drainage profiles at any time in one simple oepration. It requires only a desk top calculator to determine such profiles.

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THE SIMPLIFIED DRAINAGE MODEL

In a shallow soil layer overlaying an impermeable bed, soil water diffusion tends to maintain a uniform water content within the layer along the slope. Therefore the volumetric water content θ at any point along the slope y may be assumed to be uniform with respect to soil depth. Then the variation of θ with time t is primarily dependent on the downslope flux q, which is controlled by the hillslope angle β and the soil water hydraulic conductivity $K(\theta)$. Under these simplifying conditions the equation governing the subsurface flow of water per unit width of hillslope is given by

$$\frac{\partial \theta}{\partial t} + \frac{\partial q}{\partial y} = I \tag{1}$$

where

$$q = \sin \beta[K(\theta) - K(\theta = \theta_0)] \tag{2}$$

In general, I is a source/sink term representing the difference between evapotranspiration and precipitation, and y is directed down the slope ($y = 0$ is located at hilltop, and $y = L$ is located at the stream channel). Equation (1) ignores capillarity effects along the slope. This is in agreement with the simple drainage model developed by *Sisson et al.* [1980] and further justified and expanded by *Hurley and Pantelis* [1985]. The main difficulty with these models is that, in principle, the soil drains to a moisture content θ_r, such that $K(\theta_r) = 0$. However, capillarity may prevent this from happening. For example, in the Coweeta experiment, if capillarity is ignored, more water would be found to drain than actually did. In principle, then, the effect of capillarity must be taken into account so that, at any position y, drainage occurs until a value θ_0 is reached, as determined by the capillary rise profile. We simplify this approach by replacing $\theta_0(y)$ by its average value. The explicit calculation of the θ_0 average will be given later. Then we take the value for θ_0 into account in the definition of the flux q, in (2), so that drainage is forced to stop at $\theta = \theta_0$. It is as if a certain flux of water corresponding to $K(\theta_0)$ were artificially added to the top of the slope to compensate for the extra drainage erroneously obtained when capillarity is ignored. Although the definition of q seems to be somewhat artificial or ad hoc, the introduction of $K(\theta_0)$ does not significantly affect the numerical results, provided $K(\theta_0)$ is small. It appears primarily to stop the drainage when $\theta = \theta_0$. Note also that this approach is not substantially different from infiltration experiments when an initial moisture content θ_i is present.

In such cases, the flux is reduced by $K(\theta_i)$ in order to ensure that the initial state is steady. The only difference here is that the final state rather than the initial state is relevant.

Another important effect of capillarity, which is implicit in the present model, is the maintenance of a uniform water content across the layer at a given position. That is, the soil layer must be thin enough. *Hurley and Pantelis* [1985] have a similar requirement, except that we require θ to be essentially uniform in a horizontal plane (consistent with the final capillary rise profile) rather than their more constraining condition of θ constant perpendicular to the soil layer.

It is quite interesting that the kinematic wave theories, based on the Boussinesq equation, lead to equations very similar to (1), although they are linear and describe flow in the saturated zone and hence are more appropriate for deep rather than shallow soil layers. Extensions of the kinematic wave solution include vertical unsaturated flow [*Beven*, 1982; *Smith and Hebbert*, 1983], whereas here we consider downslope unsaturated flow.

Sloan and Moore [1984] consider an initial steady state which corresponds to steady state conditions for constant precipitation and drainage. This was achieved by applying a constant input rate until steady state conditions were reached. The system was then allowed to drain with no further input. Therefore the appropriate form of (1) for the Coweeta experiment is

$$\frac{\partial\theta}{\partial t} + Q\frac{\partial\theta}{\partial y} = 0 \qquad (3)$$

where

$$Q = \sin\beta\left(\frac{dK}{d\theta}\right) \qquad (4)$$

Equation (3) and (4) are subject to the initial condition,

$$q_i(y) = q(t = 0, y) = Iy \qquad (5)$$

The characteristics of (3) are obtained by solving the following parametric equations:

$$dt = Q^{-1}\,dy = \frac{d\theta}{0} \qquad (6)$$

The left-hand side implies that θ is constant on the line

$$\frac{dy}{dt} = Q \qquad (7)$$

Equation (7) can be integrated to give

$$y = Qt + y_0 \qquad (8)$$

Equation (7) and (8) illustrate that the characteristics are straight lines traveling with constant speed Q and issue from the initial condition q_i. Therefore the solution to the initial value problem is given by

$$q(y, t) = q_i(y_0) = I(y - Qt) \qquad (9)$$

Since q_i is an increasing function of y (equation (5)), then $q(y, t)$ is continuous for all $t \geq 0$. If q_i was not an increasing function of y, then the presence of mathematical discontinuities in the solution would lead to the development of shock waves [*Lax*, 1972].

DETERMINATION OF DRAINAGE AND CUMULATIVE DRAINAGE PROFILES

Several functional dependences of K on θ have been proposed. We will adopt the one suggested by *Brooks and Corey* [1964] to easily compare with Sloan and Moore's results,

$$K(\theta) = K_s\left(\frac{\theta - \theta_r}{\theta_s - \theta_r}\right)^{1/n} \qquad (10)$$

The value n is a soil parameter dependent on soil type, K_s is the saturated hydraulic conductivity, θ_r is the residual moisture content, and θ_s is the saturated moisture content. Other analytical forms for $K(\theta)$, e.g., an exponential dependence [*Watson*, 1967], could be used just as easily with our approach instead of (10).

The drainage rate from the soil profile D_r is given by

$$D_r = wbq(\theta_L) \qquad (11)$$

where

$$\theta_L = \theta(t, y = L) \qquad (12)$$

w is the width, and b is the average soil depth. The cumulative drainage volume M is given by

$$M = wb\int_0^L [\theta(t = 0, y) - \theta(t, y)]\,dy \qquad (13)$$

From (2), (4), (9), and (10),

$$\left(\frac{\theta - \theta_r}{\theta_s - \theta_r}\right)^{1/n} = \frac{Iy}{K_s\sin\beta} - \frac{It}{n(\theta_s - \theta_r)} \cdot \left(\frac{\theta - \theta_r}{\theta_s - \theta_r}\right)^{(1-n)/n} + \left(\frac{\theta_0 - \theta_r}{\theta_s - \theta_r}\right)^{1/n} \qquad (14)$$

Equation (14) is easily applied to obtain the time when θ has a certain value at a certain position y. Using integration by parts, (13) can be rewritten as

$$M = wb\int_0^L \theta(t = 0, y)\,dy - wb\left[L\,\theta_L - \int_{\theta_0}^{\theta_L} y\,d\theta\right] \qquad (15)$$

$$M = wbL(\theta_r - \theta_L) + \alpha\left[\frac{IL}{K_s\sin\beta} + \left(\frac{\theta_0 - \theta_r}{\theta_s - \theta_r}\right)^{1/n}\right]^{n+1} + \alpha n\left(\frac{\theta_L - \theta_r}{\theta_s - \theta_r}\right)^{(1+n)/n} + t(\theta_L)\,D_r(\theta_L) - wb\sin\beta\,(\theta_L - \theta_r)\,I^{-1}\,K(\theta_0) \qquad (16)$$

where

$$\alpha = wbK_s\sin\beta(\theta_s - \theta_r)I^{-1}/(n + 1) \qquad (17)$$

and

$$t(\theta_L) = Q^{-1}(\theta_L)[L - q(\theta_L)I^{-1}] \qquad (18)$$

For any particular value of θ_L, the drainage D_r and the cumulative drainage M corresponding to θ_L can be deduced from (11), (16), and (17). They occur at a time $t(\theta_L)$ given by (18). Accordingly, for different values of θ_L varying between θ_0 and θ_s one can construct the drainage and cumulative drainage profiles immediately from (11) and (16), respectively.

RESULTS AND DISCUSSION

The values for $w = b = 0.92$ m, $L = 13.72$ m, $\sin\beta = 0.37$, and $\theta_s = 0.49$ were documented by *Hewlett* [1961] and *Hew-*

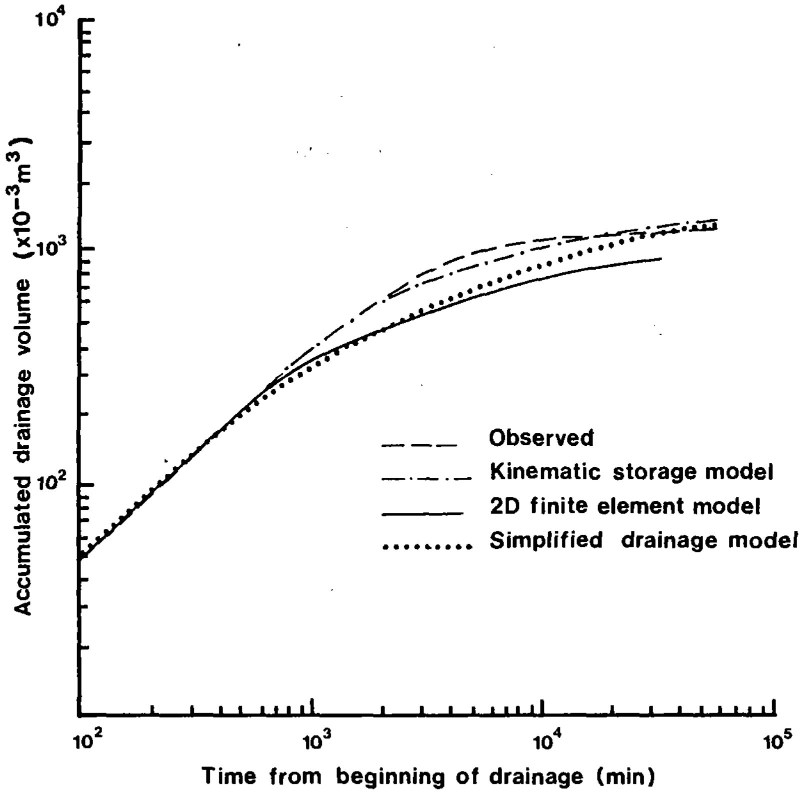

Fig. 1. Comparison between observed and predicted cumulative runoff curves for Nieber's two-dimensional finite element solution of Richard's equation, the kinematic storage model proposed by Sloan and Moore, and the simplified drainage equation. (Figure is adapted from *Sloan and Moore* [1984].)

lett and Hibbert [1963]. *Sloan and Moore* [1984] took values for $K_s = 4.032$ m/d, $\theta_r = 0.0$, and $n^{-1} = 14.63$. They also calculated the dependence of θ on the pressure head h from *Hewlett*'s [1961] experimental data and found that fitting the following model

$$\theta = (\theta_s - \theta_r)\left[\frac{A}{A + h^B}\right] + \theta_r \tag{19}$$

gave $A = 1.76$ and $B = 0.36$. The value for θ_0 represents the average of (19) for a maximum height of h^*. Therefore

$$\theta_0 = [A(\theta_x - \theta_r)/h^*] \int_0^{h^*} (A + h^B)^{-1}\, dh + \theta_r \tag{20}$$

The value for h^* is 5.0 m. Consequently, $\theta_0 = 0.282$.

The only parameter that remains unknown is I. Its value determines the shape of q_i and, consequently, the solution. *Hewlett and Hibbert* [1963] fit the following model to the early drainage phase data (for $t < 1$ d)

$$D_r = 0.333\, t^{-0.28} \tag{21}$$

As *Hewlett and Hibbert* [1963] did not provide details for $D_r(t = 0)$ and no estimate can be made from (21), some other experimental data must, therefore, be used. Sloan and Moore took a value for steady state discharge equal to 0.692 m^3/(d m). We arrive at a similar value by the following procedure: We fix the value for I so that the drainage is matched at the earliest time (0.1 day) presented in *Hewlett and Hibbert's* [1963] figure. For example, the drainage rate at 0.1 day is 0. 635 m^3/d, $q = 0.750$ m/d, and with $n^{-1} = 14.63$ and $\theta_0 = 0.282$, $Q = 23.45$ m/d and $I = 0.0659$ m/(d m). The value of I can now be used to calculate the drainage rate and cumulative drainage profiles for the Coweeta data.

From (11) and (16),

$$D_r = 4.303 \times 10^4\, \theta_L^{14.63} - 3.898 \times 10^{-4} \tag{22}$$

and the cumulative drainage is given by

$$M = 5.151 - 11.62\, \theta_L + 4.177 \times 10^4\, \theta_L^{15.63} + D_r(\theta_L)t(\theta_L) \tag{23}$$

where these values are calculated for time given by

$$t(\theta_L) = \left[\frac{13.73 - 7.714 \times 10^5\, \theta_L^{14.63}}{7.437 \times 10^5\, \theta_L^{13.63}}\right] \tag{24}$$

The cumulative drainage for the simplified drainage model is plotted in Figure 1 along with Sloan and Moore's kinematic storage model, Hewlett and Hibbert's experiment, and Nieber's finite element solution to the two-dimensional Richard's equation. Our simplified drainage model is an approximate solution to Richard's equation. Clearly, our model is in good agreement with Nieber's two-dimensional numerical solution of Richard's equation, which, in principle, should be more accurate.

Sloan and Moore's kinematic storage model assumes that the hydraulic gradient equals the slope of the impermeable bed and therefore is also an approximation of Richard's equation. Sloan and Moore's model is in worse agreement with the numerical solution but is in better agreement with the experimental observation compared with our model. Both models give a better fit to the experimental data than the, in principle, more exact two-dimensional numerical solution.

CONCLUSION

The simplified drainage model adapted from *Sisson et al.* [1980] has been demonstrated to provide a good approximation of Richard's equation when applied to the Coweeta data. The model gives reliable prediction of both short- and long-term drainage.

The one distinct advantage that the simplified drainage model exhibits over other models is that it is fully analytic and can be used to predict drainage and cumulative drainage profiles instantly and without recourse to computer algorithms. Such simplicity suggests possible future integration into existing models used for predicting watershed discharge. Models extended to watersheds and taking into account surface runoff will be presented in the near future.

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