WATER RESOURCES RESEARCH, VOL. 29, NO. 8, PAGES 2505–2514, AUGUST 1993

A Modeling Study of Rainfall Rate–Reflectivity Relationships

JAMES A. SMITH

Department of Civil Engineering and Operations Research, Princeton University, Princeton, New Jersey

WITOLD F. KRAJEWSKI

Iowa Institute of Hydraulic Research, Department of Civil and Environmental Engineering, University of Iowa, Iowa City

Power law models that relate rainfall rate and radar reflectivity factor are the principal topic of this paper. Two interrelated problems associated with these models are examined: (1) estimation of parameters of power law models, and (2) assessment of the accuracy of rainfall rate estimates derived from power law models. A statistical model of raindrop processes is used for analysis of both problems. The model provides explicit representations of power law model parameter estimates and the error of rainfall rate–reflectivity relationships in terms of simple representations of raindrop processes. These results are used to examine issues related to parameterization of algorithms for radar rainfall estimation including (1) bounds on power law model parameter estimates, (2) climatological variability of power law model parameters, (3) seasonal variability of power law model parameters, and (4) multiplicative bias in radar rainfall estimates. Empirical analyses are carried out using drop size data from a number of sites. Detailed analyses are carried out for a data set from North Carolina.

1. INTRODUCTION

The relationship between rainfall rate and radar reflectivity factor is often represented by a power law model of the form

$$R = \alpha Z^{\beta} \varepsilon \qquad (1)$$

where R is rainfall rate, Z is radar reflectivity factor, α and β are model parameters, and ε is a multiplicative error term. The principal topics of this paper concern statistical estimation of parameters of the power law model.

The parameters α and β are used in implementing rainfall estimation procedures based on radar-observed values of reflectivity (the terms "reflectivity" and "radar reflectivity factor" will be used synonymously in this paper). A bewildering range of power law parameters has been reported in the literature [see *Battan*, 1973; *Stout and Mueller*, 1968]. The sources of variability in estimates of power law model parameters are examined in this paper. Parameters of the error process ε are important for assessing accuracy of rainfall estimates derived from the power law model and observations of radar reflectivity factor. In many studies of rainfall rate–reflectivity relationships, the error process is only implicitly included in the model formulation.

Drop size data from a number of sites are used to examine rainfall rate–reflectivity relationships. The principal advantage of using drop size data for studying rainfall rate–reflectivity relationships is that paired values of rainfall rate and reflectivity can be obtained. Two differences between reflectivity data derived from surface drop size observations and reflectivity data obtained from radar should be emphasized. Radar samples a volume of approximately 1 km^3; surface drop size data represent a volume of approximately 1 m^3. Radar samples the atmosphere (to elevations exceeding 10 km); surface drop size data represent conditions within several meters of the ground surface.

Paper number 93WR00962.
0043-1397/93/93WR-00962$05.00

A simple parameterization of raindrop processes [see *Smith*, 1993] is used to examine the variability of power law parameters. The model of *Smith* [1993] yields a parameterization of rainfall rate and reflectivity in terms of the drop arrival rate, mean diameter of raindrops, and standard deviation of raindrop diameter. The model also provides a tool for examining time-varying errors in rainfall rate estimates derived from power law models. Previous studies of drop size distributions that are particularly relevant to this study include the works of *Jones* [1992], *Atlas and Chmela* [1957], *Cataneo and Stout* [1968], *Stout and Mueller* [1968], *Nicholas and Larke* [1976], *Willis and Tattleman* [1989], and *Waldvogel* [1974].

Radar rainfall estimation entails two principal steps: (1) conversion of reflectivity measured in the atmosphere to surface reflectivity and (2) conversion of surface reflectivity to rainfall rate. The first step encompasses a wide range of procedures and issues [see *Austin*, 1987; *Joss and Waldvogel*, 1989]. As indicated above, the focus of this paper is the second element of radar rainfall processing. Broader contributions are also made, principally by clarifying the interaction between the two elements of radar rainfall processing.

Radar estimates of rainfall serve an increasingly important role in hydrometeorological problems. Major hydroclimatological programs in which radar plays a prominent role include the Tropical Rainfall Measuring Mission (TRMM) [see *Simpson et al.*, 1988], the Global Energy and Water Cycle Experiment (GEWEX) Continental-Scale International Project [see *Chahine*, 1992], and the Global Precipitation Climatology Project [see *Arkin and Ardanuy*, 1989]. The Next Generation Weather Radar (NEXRAD) system, which is being deployed throughout the United States during the period 1991–1997, will have a diverse range of hydrologic applications ranging from short-term flood forecasting to water resources management [see *Hudlow et al.*, 1991]. Results of this study will aid in parameterization of algorithms used for radar rainfall estimation and in assessing the accuracy of the rainfall estimates that are obtained from radar.

2. DEFINITIONS AND NOTATION

In this section notation and models needed to relate rainfall rate and reflectivity to drop size distributions are introduced. The section begins with a brief description of the drop size data used in subsequent sections.

Observations of drop size distributions at the ground surface were made at a number of sites during the 1960s using the Illinois State Water Survey raindrop camera (for a description of the instrument, see *Jones and Dean* [1953]). The North Carolina observations [see *Jones*, 1992; *Mueller and Sims*, 1967] were taken in the Coweeta experimental watershed from 1960–1962 and consist of 4741 samples of drop size distributions. An individual drop size distribution consists of drop counts and drop diameters within a sample volume of approximately 1 m^3. The time interval between observations is 1 min. The sampling time interval arises as follows. At the beginning of each minute the raindrop camera becomes active for several seconds and takes snapshots of raindrops that cumulatively represent a 1 m^3 sample volume. The camera is then inactive for the remainder of the minute. The observations are reported as drop counts in 65 equally spaced 0.1-mm size classes ranging from 0.5 mm to 7 mm (a reported diameter represents diameter of a sphere of equivalent mass).

The observations provide a representation of drop density (in drops per cubic meter) and drop diameters at fixed points in time. The observations can be converted to drop arrival rate values (in drops per square meter per second) by utilizing the following representation for terminal velocity (in meters per second) of raindrops in still air:

$$v(D) = c_1 D^{c_2} \tag{2}$$

where D is drop diameter (in millimeters). *Atlas and Ulbrich* [1977] fit the *Gunn and Kinzer* [1949] terminal velocity data to (2), obtaining $c_1 = 3.87$ m/s and $c_2 = 0.67$.

Figure 1 illustrates sample statistics derived from the drop size data for a storm that occurred on December 21, 1960, in North Carolina. The figure shows the drop arrival rate, sample mean of the drop diameters, and coefficient of variation of the drop diameters. Note the pronounced temporal variability in each process. The distribution of drop diameters over the 2-year observing period is illustrated in Figure 2. The mode of the distribution is at 0.9 mm. The distribution falls off sharply for small drops and very large drops. Small drops may be underrepresented by the raindrop camera [*Rinehart*, 1983]. This will have little impact on conclusions concerning rainfall rate and reflectivity, because of the high-order power law dependence on drop diameter (see development in the appendix).

Rainfall rate and reflectivity can be defined in terms of a marked point process model that represents the arrival times and diameters of raindrops (see development in the appendix). The model of *Smith* [1993] for drop size distributions is used in analyzing the relationship between rainfall rate and reflectivity. The model can be described as follows. Let $R(t)$ and $Z(t)$ denote rainfall rate (millimeters per hour) and reflectivity (mm^6 m^{-3}) at time t of a storm and let $\lambda(t)$, $\mu(t)$, and $\sigma(t)$ be stochastic processes with the following interpretation: $\lambda(t)$ is the mean number of drop arrivals at the top of the sample volume (in drops per square meter per second); $\mu(t)$ is the mean of the natural logarithm of drop diameter; and $\sigma(t)$ is the standard deviation of the log diameter of drops. The model for drop arrivals and drop diameters is specified by the assumptions that (1) the drop arrival process is a Poisson process with randomly varying rate of occurrence $\lambda(t)$ and (2) drop diameters arriving at the top of the sample volume have a lognormal distribution with randomly varying parameters ($\mu(t)$, $\sigma(t)$) (see the appendix for additional details).

The accuracy of the lognormal assumption for the distribution of drop diameters is illustrated in Figure 3, which shows a lognormal quantile-quantile plot for scaled drop diameters. The logarithm of drop diameter is scaled by subtracting the sample mean and dividing by the sample standard deviation of the logarithm of drop diameters for the 1-min sample. Figure 3 indicates that the lognormal assumption is quite reasonable for the central portion of the distribution. It does not help in deciding whether possible departures from the lognormal assumption for small and large drops seriously affect the capability for representing rainfall rate and reflectivity. This issue, however, can be addressed directly.

The following results relating the raindrop parameterization ($\lambda(t)$, $\mu(t)$, $\sigma(t)$) to rainfall rate and reflectivity can be derived from results of *Smith* [1993]:

$$R(t) = [6\pi \times 10^{-4}]\lambda(t) \exp\{3\mu(t) + 4.5\sigma(t)^2\} \tag{3}$$

$$Z(t) = \frac{\lambda(t)}{c_1} \exp\left\{(6 - c_2)\mu(t) + \frac{1}{2}(6 - c_2)^2\sigma(t)^2\right\} \tag{4}$$

The relationships are quite accurate (see Figure 4). Observed values of rainfall rate and reflectivity are computed from drop size observations using equations (A3) and (A4) of the appendix. Model-derived estimates of rainfall rate are based on estimators of the time-varying parameters ($\lambda(t)$, $\mu(t)$, $\sigma(t)$). The estimators are computed from sample statistics of the drop size data illustrated in Figure 1. Comparisons of the lognormal model with a variety of models commonly used in the radar meteorology literature are given by *Smith* [1993].

3. ESTIMATION OF POWER LAW MODEL PARAMETERS

Battan [1973] presents an extensive summary of power law model parameters obtained from drop size data around the world. Exponents of the power law model range from 0.34 [see *Higgs*, 1952] to 0.87 [see *Atlas and Chmela*, 1957]. In this section, the raindrop parameterization introduced in the previous section is used to analyze the variability of estimated power law model parameters. Both a deterministic power law model and a statistical power law model are examined.

A deterministic power law model takes the form

$$R(t) = \alpha Z(t)^\beta \tag{5}$$

Equations (3) and (4) imply that the following conditions are required for a deterministic power law relation between Z and R to hold:

$$\lambda(t) = \lambda(t)^\beta \tag{6}$$

$$3\mu(t) = \beta(6 - c_2)\mu(t) \tag{7}$$

$$4.5\sigma(t)^2 = \beta\,\tfrac{1}{2}(6 - c_2)^2\sigma(t)^2 \tag{8}$$

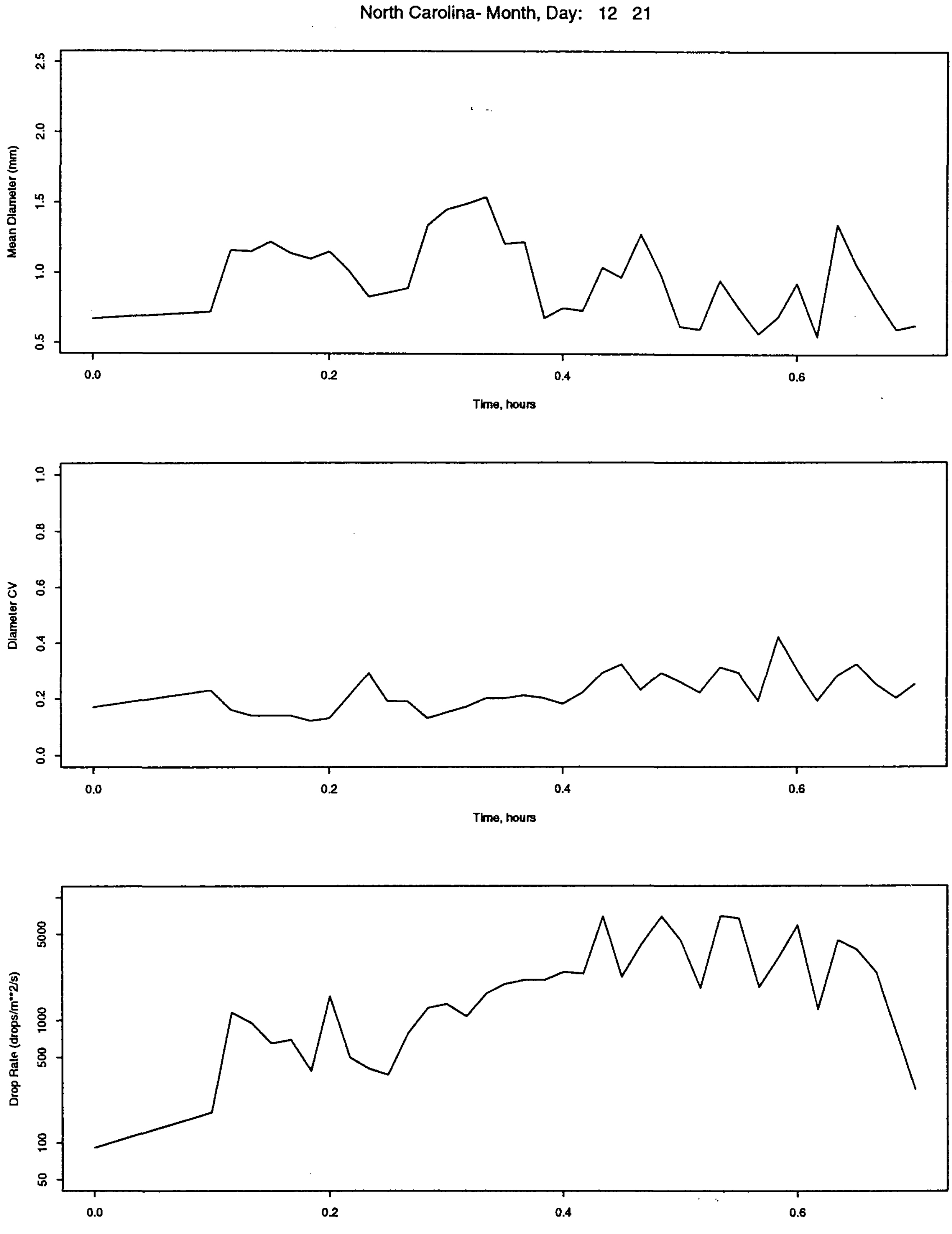

Fig. 1. Time series plots of mean diameter (in millimeters), coefficient of variation of drop diameter, and drop arrival rate (in drops per square meter per second) for a storm in North Carolina on December 21, 1960.

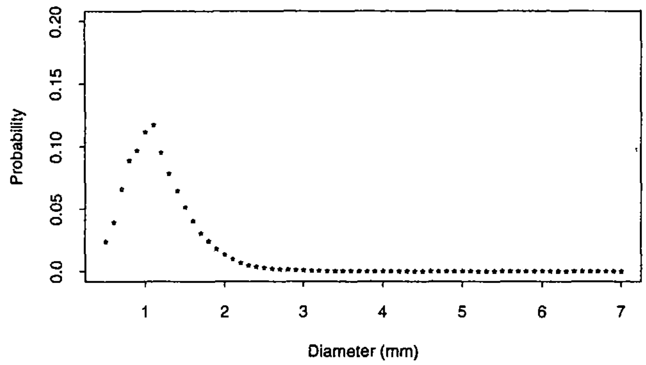

Fig. 2. Histogram of drop diameters (in millimeters) for the North Carolina drop size data.

It follows from (6) that

$$\beta = 1 \tag{9}$$

Equation (7) (with $c_2 = 0.67$) implies that

$$\beta = 0.56 \tag{10}$$

Equation (8) (with $c_2 = 0.67$) implies that

$$\beta = 0.32 \tag{11}$$

These results illustrate two points about rainfall rate–reflectivity relationships. The first point is the well-known fact that there is not a unique relationship between rainfall rate and reflectivity. Consequently, a statistical model relating rainfall rate and reflectivity is required. The second point is that the raindrop parameterization provides a quantitative tool for characterizing the dependence of power law model parameters on raindrop processes. The results suggest that power law model parameters can range from 0.32 to 1.0 and that simple raindrop characterizations determine the bounds (more on this below). It is notable that the range of power law exponents obtained from the preceding analysis corresponds closely to the range of empirical values reported by *Battan* [1973].

Although there is not a unique relationship between rainfall rate and reflectivity, there is generally a strong statistical relationship (see, for example, Figure 5). A statistical power law model takes the form

$$R(t) = \alpha Z(t)^{\beta} \varepsilon(t) \tag{12}$$

where $\varepsilon(t)$ is a multiplicative error process and (α, β) are unknown parameters. The standard deviation of the error process is denoted s, i.e.,

$$s = \mathrm{Var}\,(\varepsilon(t))^{1/2} \tag{13}$$

Drop size data provide rainfall rate and reflectivity observations at discrete points in time rather than continuously in time. The sample observations of rainfall rate and reflectivity obtained from n drop size samples will be denoted (R_i, Z_i), $i = 1, \cdots, n$. The power law model for the sample observations is denoted by

$$R_i = \alpha Z_i^{\beta} \varepsilon_i \tag{14}$$

Under the lognormal model for the error process,

$$\ln \varepsilon_i \sim N(0, w^2) \tag{15}$$

i.e., the logarithm of the error process has a normal distribution with mean 0 and variance w^2. The log variance parameter w^2 can be expressed in terms of the parameter s as follows:

$$w^2 = \ln\left(\frac{1 + [1 + 4s^2]^2}{2}\right) \tag{16}$$

Given a sample of paired rainfall rate and reflectivity values (R_i, Z_i), $i = 1, \cdots, n$, the power law parameters can be estimated by linear regression of ln (R_i) versus ln (Z_i), yielding

$$\hat{\beta} = \frac{\sum_{i=1}^{n} (\ln Z_i - \overline{\ln Z})(\ln R_i - \overline{\ln R})}{\sum_{i=1}^{n} (\ln Z_i - \overline{\ln Z})^2} \tag{17}$$

$$\hat{\alpha} = \exp\,\{\overline{\ln R} - \hat{\beta}\,\overline{\ln Z}\} \tag{18}$$

where

$$\overline{\ln Z} = n^{-1} \sum_{i=1}^{n} \ln Z_i \tag{19}$$

$$\overline{\ln R} = n^{-1} \sum_{i=1}^{n} \ln R_i \tag{20}$$

The raindrop parameterization can be used together with (17) to examine sources of variability in estimating power law model parameters. Substituting the representations for rainfall rate and reflectivity in terms of drop distribution parameters (equations (3) and (4)) into the regression relationship of (17), the following representation for the estimator of β is obtained:

$$\hat{\beta} = \frac{\sum_{i=1}^{n} A_i B_i}{\sum_{i=1}^{n} A_i^2} \tag{21}$$

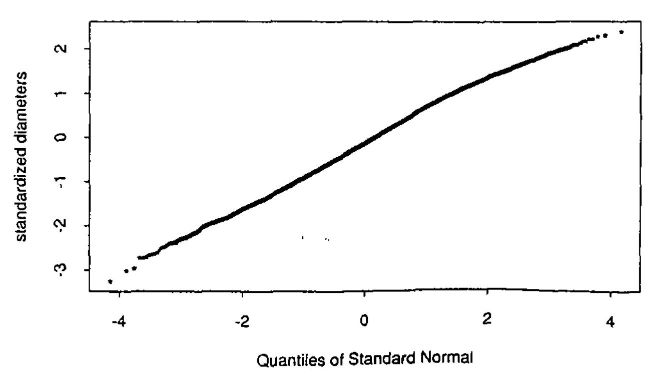

Fig. 3. Lognormal quantile-quantile plot for scaled drop diameters.

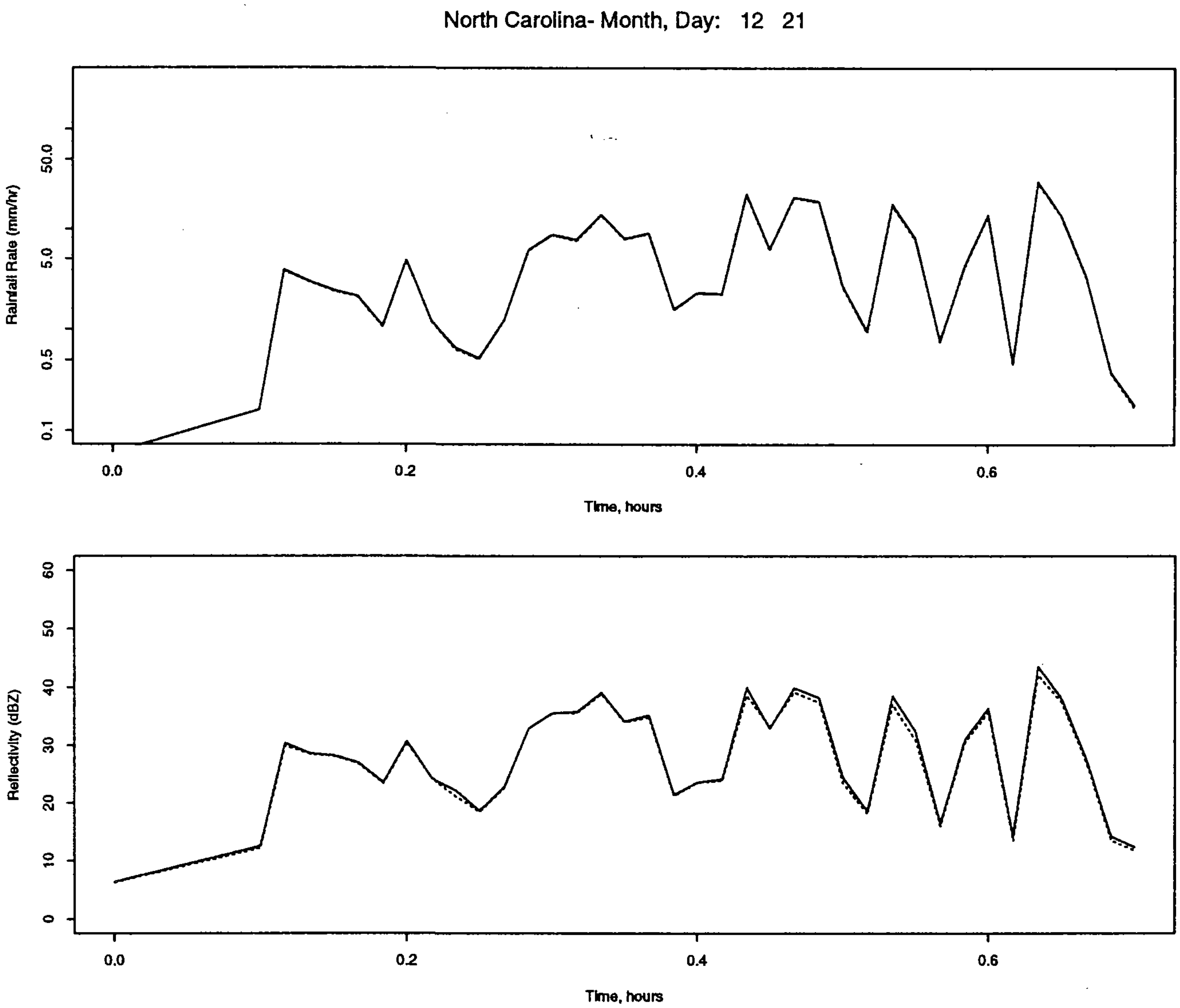

Fig. 4. Model estimates (solid lines) and observations (dashed lines) of rainfall rate and reflectivity for the December 21 storm.

where

$$A_i = \sum_{i=1}^{n} [\ln(\lambda_i) - \bar{\lambda}] + (6 - c_2)(\mu_i - \bar{\mu}) + \tfrac{1}{2}(6 - c_2)^2(\sigma_i^2 - \bar{\sigma}^2) \quad (22)$$

$$B_i = \sum_{i=1}^{n} [\ln(\lambda_i) - \bar{\lambda}] + 3(\mu_i - \bar{\mu}) + 4.5(\sigma_i^2 - \bar{\sigma}^2) \quad (23)$$

and $\bar{\lambda}$ is the mean of log arrival rate values:

$$\bar{\lambda} = n^{-1} \sum_{i=1}^{n} \ln \lambda_i \quad (24)$$

The principal conclusion that is obtained from (21) is that power law parameter estimates are determined by the relative variability of drop arrival rate and drop diameter properties. This point is examined below from two perspectives.

If there is no variability in drop arrival rate, i.e., if $\ln(\lambda_i) = \bar{\lambda}$, and no variability in the log variance of the drops, i.e., if $\sigma_i = \bar{\sigma}$, then (21) reduces to (10) and the estimate of β is 0.56. Similarly, if there is no variability in drop arrival rate and log mean of the drop diameters, (21) reduces to (11) and the estimate is 0.32. If there is no variability in drop diameter characteristics, i.e., if $\mu_i = \bar{\mu}$, $\sigma_i = \bar{\sigma}$, then (21) reduces to (9) and the estimate is 1.0. Thus for the extreme case in which variability of one process (among drop arrival rate, mean log diameter, and standard deviation of log diameter) is dominant, the estimate of the power law exponent is determined by the relationships developed at the beginning of the section. These results illustrate that least squares estimates of the power law exponent are bounded below by 0.32 and above by 1.0 and that the bounds are determined by specific raindrop processes.

A clearer picture can be obtained by simplifying (21). If

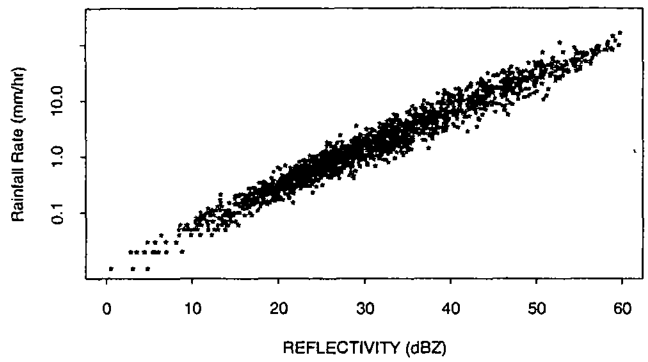

Fig. 5. Scatterplot of rainfall rate and reflectivity observations obtained from the North Carolina drop size data.

TABLE 1. Estimated Power Law Parameters for 10 Sites

	β	α	s	ρ	Number of Samples
Marshall Islands	0.76	0.016	0.39	0.96	2659
Panama	0.75	0.012	0.37	0.98	3359
Florida	0.74	0.013	0.37	0.97	2501
North Carolina	0.72	0.019	0.39	0.97	4741
New Jersey	0.71	0.018	0.36	0.96	3103
Illinois	0.70	0.015	0.37	0.97	1717
Indonesia	0.70	0.017	0.34	0.98	1872
Arizona	0.67	0.013	0.36	0.98	1949
Alaska	0.64	0.026	0.31	0.95	2686
Oregon	0.62	0.028	0.31	0.92	1703

The parameter ρ is the correlation between log rainfall rate and log reflectivity. The right column gives the total number of drop size samples used in analyses for the site.

cross-product terms in (21) are neglected (cross-correlation values for raindrop processes are less than 0.25), the estimator for β can be represented as follows:

$$\hat{\beta} \approx \frac{1 + 16I_1 + 62.1I_2}{1 + 28.4I_1 + 190I_2} \tag{25}$$

where

$$I_1 = \frac{\sum_{i=1}^{n} (\mu_i - \bar{\mu})^2}{\sum_{i=1}^{n} (\ln \lambda_i - \bar{\lambda})^2} \tag{26}$$

$$I_2 = \frac{\sum_{i=1}^{n} (\sigma_i^2 - \bar{\sigma}^2)^2}{\sum_{i=1}^{n} (\ln \lambda_i - \bar{\lambda})^2} \tag{27}$$

The quantities I_1 and I_2 can be interpreted as indexes of the relative variability of drop arrival rate and drop diameter characteristics. Estimates of the power law exponent will be small if one or more of the indices is large. If both indices are close to 0, the estimate of the power law exponent will be close to 1.

Table 1 summarizes power law parameter estimates for 10 sites representing a broad range of climatic conditions. The estimated values of β range from a minimum of 0.62 for Corvallis, Oregon, to a maximum of 0.76 for Majuro in the Marshall Islands. The conclusions drawn from (21) and Table 1 are qualitatively similar to those obtained by other investigators. *Stout and Mueller* [1968, p. 469], for example, attribute the difference in power law exponents between Alaska and the Marshall Islands to the fact that "drop size spectra in the Marshall Islands contain a relatively large number of small droplets ⋯". There are two distinctive features of the results obtained from the present analysis. First, specific quantitative measures are used to summarize raindrop processes (equations (22), (23), (26) and (27)). Second, these summary quantities lead to explicit quantitative relationships with power law parameters (equations (21) and (25)).

An important source of uncertainty that arises in relating the preceding results to radar rainfall processing is associated with scale effects arising from the different sampling scales of radar and the raindrop camera. One method for more closely relating drop size observations to radar sampling is to average in time, i.e., trade time for space (see, for example, *Zawadzki* [1975]). Figure 6 shows power law estimates for North Carolina obtained from drop size samples averaged over 1- to 60-min time periods. The estimated power law exponent drops from 0.72 to 0.68 at approximately 30-min time averages and increases slightly thereafter. These results suggest that corrections for scale effects should be considered, but that the magnitudes of scale effects are significantly smaller than the range of climatic variability.

Estimates of the standard deviation parameter s are used to assess the accuracy of rainfall rate estimates derived from reflectivity observations. Estimated values of the parameter s range from 0.31 to 0.39 (Table 1) with smallest values at the low–rainfall rate sites and largest values at the high–rainfall rate sites. The estimates of s, together with (18), can be used to determine the interval containing 67% of the error values for each site. For $s = 0.31$ (Alaska and Oregon) the range is 0.73–1.36, i.e., 27% underestimation to 36% overestimation. For $s = 0.39$ (North Carolina and Marshall Islands), the range is 0.68–1.48. The large error bounds that are associated with rainfall rate estimates have been a strong motivation for developing time-varying rainfall rate–reflectivity

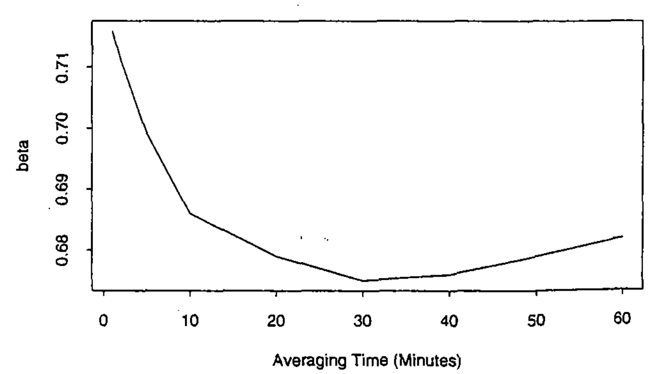

Fig. 6. Dependence of the estimated power law exponent on averaging time (North Carolina).

relationships. This topic is examined in detail in the following section.

The errors in estimating rainfall from radar will result from a combination of errors associated with the conversion of reflectivity measured aloft to reflectivity at the ground and the conversion of reflectivity to rainfall rate. The preceding results illustrate the magnitude of the second component of measurement error for radar rainfall estimates.

4. TEMPORAL VARIABILITY OF RAINFALL RATE-REFLECTIVITY RELATIONSHIPS

Many authors have noted that rainfall rate-reflectivity relationships vary not only from site to site but also with time at a single site (see *Battan* [1973] for a summary). Two aspects of temporal variation of rainfall rate-reflectivity relationships are examined in this section: (1) the magnitude and causes of storm-to-storm variability in power law parameters and (2) the effects of storm-to-storm variability in power law parameters on the error characteristics of rainfall rate estimates derived using a fixed power law model.

The temporal component of the modeling framework will be modified in this section by splitting rainfall rate and reflectivity observations into storms. Two sampling models will be used. In the first model, storm-to-storm variability in power law parameters is allowed:

$$R_{ij} = \alpha_i Z_{ij}^{\beta_i} \varepsilon_{ij} \tag{28}$$

Here, R_{ij} is rainfall rate for the jth minute of the ith storm and z_{ij} is the corresponding reflectivity observation. In the second model the power law parameters do not vary from storm to storm, but storm-to-storm variability in the error process will be maintained, i.e.,

$$R_{ij} = \alpha Z_{ij}^{\beta} \varepsilon_{ij} \tag{29}$$

For empirical analyses, storms are taken to be periods of continuous 1-min observations with a minimum of 20 observations. For the North Carolina data there are 80 storms during the 2-year period of record.

For the sampling model of (28), power law parameters were estimated for each of the 80 storms. Figure 7 illustrates the joint distribution of $(\hat{\alpha}_i, \hat{\beta}_i)$. As dictated by (18), the estimates of the two power law parameters are highly correlated.

Estimates of the exponent range from 0.5 to near 1.0, a range much larger than that associated with climatological

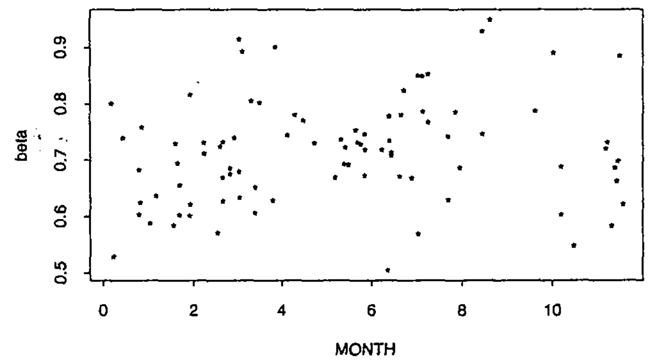

Fig. 8. Sample estimates of β versus time of year for storms in North Carolina (January is month 1).

variability in power law exponents. Although a portion of this variability may be attributed to small-sample properties of the estimators, there is a strong indication for pronounced storm-to-storm variability in power law exponents. Possible sources of storm-to-storm variability in power law parameters are examined below. Of particular interest are sources of variability that could be identified by information that is operationally available for radar rainfall processing (in the NEXRAD system, for example, information is limited to rain gage data and full volume scan reflectivity data).

Operational radar rainfall processing algorithms often accommodate seasonally varying power law parameters. In the Radar Data Processor, version 2 (RADAP II) system, for example, parameters are developed to stratify seasons in which convective precipitation dominates and seasons in which stratiform precipitation dominates [see *McDonald and Saffle*, 1989]. Figure 8 indicates that there is little relationship between time of year and estimated values of the power law exponent.

In analyzing differences between radar and rain gage observations of rainfall, *Austin* [1987] notes that many of the physical factors that result in differences depend largely on the "extent and vigor of convective activity." Maximum reflectivity is used by Austin as a tool for assessing the extent and vigor of convective activity. For the North Carolina drop size data, Figure 9 illustrates the relationship between maximum reflectivity for a storm and the estimated value of β_i for the storm. Once again, there is little or no relationship between the two.

As *Austin* [1987] demonstrates, convective activity is an important variable for quantifying errors in radar rainfall

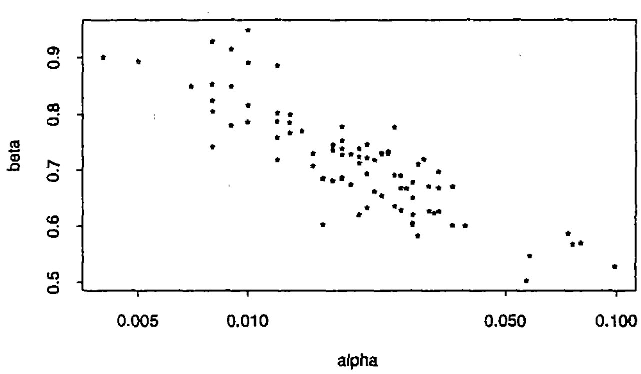

Fig. 7. Estimates of the power law parameters α and β for storms in North Carolina.

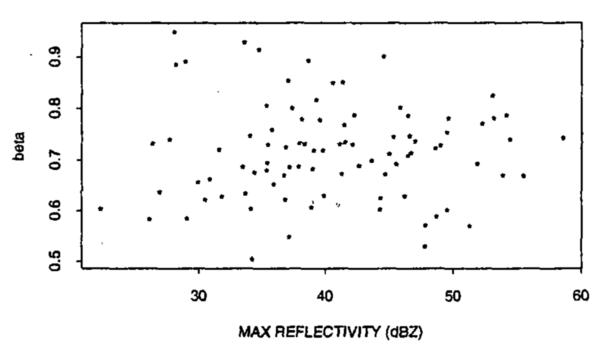

Fig. 9. Sample estimates of β versus maximum storm reflectivity for storms in North Carolina.

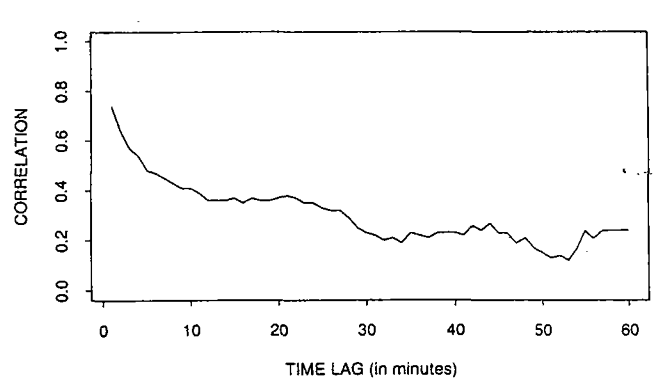

Fig. 10. Sample autocorrelation function for the error process ε_{ij} of (30).

estimates associated with radar sampling (incomplete beam filling, bright band, updrafts/downdrafts, presence of hail, etc.). These errors, however, are largely ones in which radar estimates of reflectivity aloft are not representative of reflectivity at the surface (see also *Joss and Waldvogel* [1989], *Collier* [1989], and *Zawadzki* [1982]). Figures 8 and 9 suggest that, unlike the algorithms associated with conversion of reflectivity aloft to reflectivity at the surface, seasonal variation is not a major source of variability in power law models relating reflectivity and rainfall rate.

The error characteristics of rainfall rate estimates derived from fixed power law models are examined below. Attention switches from the sampling model of (28) to the sampling model of (29).

The drop size parameterization can be used to assess the error of rainfall rate estimates based on (29). The error in the estimate can be evaluated as follows using the representations for rainfall rate and reflectivity (equations (3) and (4)), $c_2 = 0.67$, and the estimated North Carolina parameters, $\alpha = 0.019$ and $\beta = 0.72$,

$$\varepsilon_{ij} = 0.30\lambda_{ij}^{0.28} \exp\{-0.7\mu_{ij} - 4.5\sigma_{ij}^2\} \qquad (30)$$

Underestimation results when the drop arrival rate is relatively large compared to drop diameter characteristics. Conversely, overestimation results when drop diameter statistics are large relative to drop arrival rate. Note that the situation is analogous to the parameter estimation problem. Large values of drop arrival rate do not necessarily lead to underestimates of rainfall rate, nor do small values necessarily lead to overestimates. The magnitude of errors depends on the relative magnitudes of drop arrival rate and drop diameter statistics.

Equation (30) suggests another feature of estimation error; errors are characterized by temporal correlation. From (30) it follows that the correlation of errors of rainfall rain estimates is directly related to the temporal correlation of drop size properties. Figure 10 shows the sample autocorrelation function of model errors for the sampling model of (29). Correlation drops from 0.76 at a 1-min time lag to approximately 0.4 at a 10-min time lag.

Two pairs of rainfall rate estimates are compared in Figure 11 for a single storm, estimates derived using (28) (i.e., the sampling model that accommodates storm-to-storm variability in power law parameters) and estimates derived using (29) (i.e., the sampling model with fixed power law parameters; $\alpha = 0.019$ and $\beta = 0.72$). Figure 11 illustrates that the rainfall estimates using the long-term parameters exhibit not only autocorrelation in their errors but also a persistent multiplicative bias. Multiplicative bias can be a pervasive feature of radar rainfall estimates, leading to the necessity for adjustment algorithms that utilize rain gage data (see *Hudlow et al.* [1991] for a discussion of implications for NEXRAD algorithm design; see also *Collier* [1986] and *Smith and Krajewski* [1991]).

5. SUMMARY AND CONCLUSIONS

A simple parameterization of raindrop processes has been used to examine rainfall rate–reflectivity relationships. The principal problems considered in this paper are (1) estimation of power law parameters relating rainfall rate and reflectivity, (2) variability of estimated power law parameters, both geographically and temporally, and (3) characterization of the error of rainfall rate estimates derived from power law models and observations of reflectivity.

Empirical analyses are carried out for ground-based drop size data from 10 sites obtained using the Illinois State Water Survey raindrop camera. Extensive analyses are carried out for drop size data from North Carolina. The analyses consider the sampling properties related to natural variability of drop arrival and size distribution processes.

An analytical representation for the estimated power law exponent is developed in terms of raindrop processes. This representation is used to infer that the principal control on power law parameters is the relative variability of drop arrival rate and drop diameter characteristics. It is shown that estimated power law exponents are bounded below by 0.32 and above by 1.0 and that bounds are determined by specific raindrop processes. Scale effects are examined through time averaging of drop size samples and shown to be relatively small compared with climatological variability in power law parameters.

Storm-to-storm variability in power law exponents is examined, through both empirical and theoretical analyses. For the North Carolina data set, estimated power law exponents range from 0.5 to near 1.0, a range much larger than that associated with climatological variability. It is concluded that, unlike processes that control the relationship between reflectivity aloft and surface reflectivity, the dominant controls of rainfall rate–reflectivity relationships do not have a strong seasonal component. An analytical representation for the error in rainfall rate estimates in terms

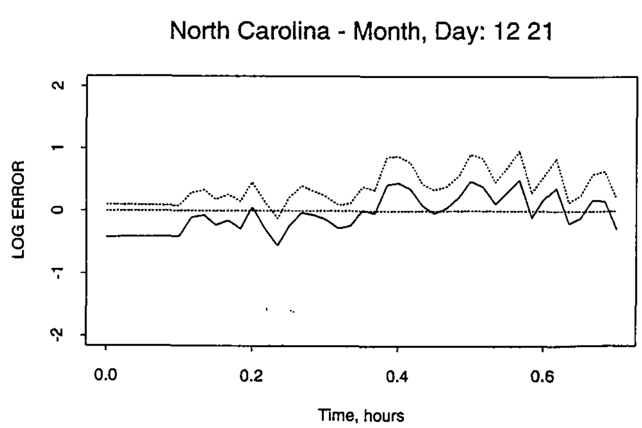

Fig. 11. Time series plots of rainfall estimate errors for the model with storm-varying power law parameters (solid line) and for the model with fixed power law parameters (dashed line).

of raindrop processes is developed. The result is similar to that arrived at for estimates of the power law exponent in that errors in the rainfall rate estimate depend solely on the relative variability of drop arrival rate and drop diameter characteristics.

Development of algorithms that utilize rain gage data to correct for systematic biases in radar rainfall estimates has been an active area of recent research (see, for example, numerous papers in the volume edited by *Cluckie and Collier* [1991]). Results of this paper illustrate two mechanisms that lead to systematic errors in radar rainfall estimates, namely, storm-to-storm variability in power law parameters and temporal correlation in raindrop processes during a storm. Bias adjustment algorithms should account for both storm-to-storm variability in bias and temporal correlation in bias.

An important area of future research is linking error characteristics of the rainfall rate–reflectivity conversion with error characteristics associated with conversion of reflectivity aloft to surface reflectivity. *Austin* [1987] provides a model for studies of this type.

APPENDIX

The model developed below represents raindrop processes continuously in time from an arbitrary time origin. The model represents fluxes of raindrops through a cubic sample element with a total volume of 1 m^3.

Let T_i denote the arrival time of the ith drop (in seconds) at the top of the sample volume. Denote the diameter of the drop (in millimeters) by D_i. The number of drops that pass through the upper surface of the sample volume up until time t is denoted by

$$\eta(t) = \sum_{i=1}^{\infty} 1(T_i \le t) \quad \text{(A1)}$$

where

$$1(T_i \le t) = 1 \qquad T_i \le t,$$

$$1(T_i \le t) = 0 \qquad T_i > t.$$

Rainfall accumulation (in millimeters) can be represented in terms of the drop arrival process through computations of the water volume of raindrops (and assuming that raindrops are perfect spheres):

$$A(t) = [(\pi/6)10^{-6}] \sum_{i=1}^{\eta(t)} D_i^3 \quad \text{(A2)}$$

Rainfall rate (in millimeters per hour) is defined relative to an accumulation time interval Δt (in seconds). We have

$$R(t) = 3600 \frac{[A(t + \Delta t) - A(t)]}{\Delta t}. \quad \text{(A3)}$$

An accumulation time interval of approximately 1 s is used for analyses presented in this paper.

Radar reflectivity factor is the sum of the sixth power of drop diameters divided by the sample volume. For a 1-m^3 sample volume, it can be represented as follows:

$$Z(t) = \sum_{i=1}^{\eta(t)} D_i^6 1(T_i + v(D_i)^{-1} > t) \quad \text{(A4)}$$

The indicator function in (A4) is used to count the number of raindrops that have arrived at the upper surface by time t, but that have not completed the trip through the sample volume by time t. As noted above, the term reflectivity will be used synonymously with radar reflectivity factor.

The model for drop arrivals and drop diameters is specified by the assumptions that (1) the drop arrival process is a Poisson process with randomly varying rate of occurrence $\lambda(t)$ and (2) drop diameters arriving at the top of the sample volume have a lognormal distribution with randomly varying parameters $(\mu(t), \sigma(t))$, that is,

$$P\{D_i \le x | \lambda(u), \mu(u), \sigma(u); u \le T_i\} = \int_0^x f(y|\mu(T_i), \sigma(T_i))\, dy \quad \text{(A5)}$$

where

$$f(y|\mu(t), \sigma(t)) = y^{-1}(2\pi)^{-1/2}\sigma(t)^{-1} \exp\left(-\frac{[\log(y) - \mu(t)]^2}{2\sigma(t)^2}\right) \quad \text{(A6)}$$

is the lognormal density function with time-varying parameters $(\mu(t), \sigma(t))$.

Acknowledgments. This research was funded by the NOAA Climate and Global Change Program under grant NA16RC0444-01. This support is gratefully acknowledged.

REFERENCES

Arkin, P., and P. Ardanuy, Estimating climatic-scale precipitation from space: A review, *J. Clim.*, *2*(11), 1229–1238, 1989.

Atlas, D., and A. C. Chmela, Physical-synoptic variations of raindrop size parameters, paper presented at Sixth Weather Radar Conference, Am. Meteorol. Soc., Cambridge, Mass., 1957.

Atlas, D., and C. Ulbrich, Path- and area-integrated rainfall measurements by microwave attenuation in the 1–3 cm band, *J. Appl. Meteorol.*, *16*, 1322–1331, 1977.

Austin, P. M., Relation between measured radar reflectivity and surface rainfall, *Mon. Weather Rev.*, *115*, 1053–1070, 1987.

Battan, L. J., *Radar Observation of the Atmosphere*, revised ed., University of Chicago Press, Chicago, Ill., 1973.

Cataneo, R., and G. Stout, Raindrop-size distributions in humid continental climates, and associated rainfall rate-radar reflectivity relationships, *J. Appl. Meteorol.*, *7*, 901–907, 1968.

Chahine, M., GEWEX: The Global Energy and Water Cycle Experiment, *Eos Trans. AGU*, *73*(2), 9–14, 1992.

Cluckie, I., and C. Collier (Eds.), *Hydrological Applications of Weather Radar*, Ellis Horwood, Chichester, England, 1991.

Collier, C. G., Accuracy of rainfall estimates by radar, I, Calibration by telemetering raingages, *J. Hydrol. Amsterdam*, *93*, 207–223, 1986.

Collier, C. G., *Applications of Weather Radar Systems*, 294 pp., Ellis Horwood, Chichester, England, 1989.

Gunn, R., and G. Kinzer, The terminal velocity of fall for water droplets in stagnant air, *J. Meteorol.*, *6*, 243–248, 1949.

Higgs, A. J., The measurement of precipitation by radar, paper presented at Third Weather Radar Conference, Am. Meteorol. Soc., Montreal, Que., Canada, 1952.

Hudlow, M. D., J. A. Smith, R. C. Shedd, and M. L. Walton, NEXRAD—New era in hydrometeorology, in *Hydrological Ap-*

plications of Weather Radar, edited by I. Cluckie and C. Collier, pp. 602–612, Ellis Horwood, Chichester, England, 1991.

Jones, D.M.A., Raindrop spectra at the ground, *J. Appl. Meteorol.*, *31*, 1219–1225, 1992.

Jones, D.M.A., and L. A. Dean, A raindrop camera, *Res. Rep. 3*, U.S. Army contract DA-36-039 SC-42446, Ill. State Water Surv., Urbana, 1953.

Joss, J., and A. Waldvogel, Precipitation measurement and hydrology, in *Radar Meteorology*, edited by D. Atlas, pp. 577–606, American Meteorological Society, Boston, Mass., 1989.

McDonald, M., and R. Saffle, RADAP II archive data user's guide, *TDL Off. Note 89-2*, U.S. Dep. of Commer., Natl. Oceanic and Atmos. Admin./Natl. Weather Serv., Tech. Dev. Lab., Silver Spring, Md., 1989.

Mueller, E. A., and A. L. Sims, Raindrop distributions at Franklin, North Carolina, *Res. Rep. 3*, U.S. Army contract DA-28-043 AMC-02071(E), Ill. State Water Surv., Urbana, 1967.

Nicholas, C., and P. Larke, Raindrop size distribution in a hilly region, *Meteorol. Mag.*, *105*, 361–381, 1976.

Rinehart, R. E., Out-of-level instruments: Errors in hydrometeor spectra and precipitation measurements, *J. Appl. Meteorol.*, *22*, 1404–1410, 1983.

Simpson, J., R. Adler, and G. North, A proposed Tropical Rainfall Measuring Mission (TRMM), *Bull. Am. Meteorol. Soc.*, *69*, 278–295, 1988.

Smith, J. A., Marked point process models of raindrop size distributions, *J. Appl. Meteorol.*, *32*, 284–296, 1993.

Smith, J. A., and W. F. Krajewski, Estimating the mean field bias of radar rainfall estimates, *J. Appl. Meteorol.*, *30*, 397–412, 1991.

Stout, G. E., and E. A. Mueller, Survey of relationships between rainfall rate and radar reflectivity in the measurement of precipitation, *J. Appl. Meteorol.*, *7*, 465–474, 1968.

Waldvogel, A., The N_0 jump of raindrop spectra, *J. Atmos. Sci.*, *31*, 1067–1078, 1974.

Willis, P., and P. Tattleman, Drop-size distributions associated with intense rainfall, *J. Appl. Meteorol.*, *28*, 3–15, 1989.

Zawadzki, I., On radar-raingage comparison, *J. Appl. Meteorol.*, *14*, 1430–1436, 1975.

Zawadzki, I., The quantitative interpretation of weather radar measurements, *Atmos. Ocean*, *20*(2), 158–180, 1982.

W. F. Krajewski, Iowa Institute of Hydraulic Research, Department of Civil and Environmental Engineering, University of Iowa, Iowa City, IA 52242.

J. A. Smith, Department of Civil Engineering and Operations Research, Princeton University, Princeton, NJ 08544.

(Received August 10, 1992;
revised March 29, 1993;
accepted April 7, 1993.)