Coniferous Stands Characterized with The Weibull Distribution

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The Weibull distribution, $F(x) = 1 - e^{-(x/\alpha)^c}$, summarized diameter, basal area, surface area, biomass, and crown profile distribution data well for several different ages of white and loblolly pine plantations. The data for diameter, basal area, surface area, and biomass were easily summarized by this one distribution in a theoretically consistent fashion. This is not possible with the normal and the gamma distributions, and the lognormal gives less satisfactory results. The distribution function should prove useful in modeling tree stands since only the parameter values need to be changed over time for the above variables. The change in these parameters may be a good way to characterize and interpret changes in stands over time.

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La fonction de Weibull, $F(x) = 1 - e^{-(x/\alpha)^c}$ résume les distributions de diamètres, surface terrière, surface de cime, biomasse et profil de couronne pour des plantations de *Pinus strobus* et *Pinus taeda* de différents âges. Les données provenant des diamètres, surface terrière, surface de cime, biomasse, se résument facilement de façon théorique et constante par cette seule distribution. Ceci n'est pas possible avec les fonctions de distribution normale et gamma, tandis que la fonction log normale donne des résultats insatisfaisants. La fonction de distribution devrait être utile pour représenter adéquatement des peuplements puisque seules les valeurs des paramètres doivent être changées en fonction du temps pour les variables déjà mentionnées. Le changement de ces paramètres peut être une bonne façon de caractériser et d'interpréter les variations observées dans les peuplements en fonction du temps. [Traduit par le journal]

Models of physical characteristics of trees and stands help to estimate growth and yield of extensive forest populations. Accurate quantification of tree characteristics permits study of the interaction among physical and physiological processes and growth. For example, quantification of diameter distributions over time allows the researcher to relate the parameters of the distribution to site, age, or stand density (Clutter and Allison 1974). He can then discover what size of trees in a given stand benefit most from such management practices as thinning, fertilization, and pruning. In many instances, these models require information on the spatial distribution of foliage (Waggoner *et al.* 1969; Murphy and Knoerr 1972; Williams and Kwi 1967; Wilson 1967).

In this paper we evaluate and develop statistical distribution models for tree diameter, basal area, surface area, and biomass, and for the vertical distribution of crown biomass and surface area in white pine (*Pinus strobus* L.) and loblolly pine (*Pinus taeda* L.) stands. The Weibull distribution is the most promising of the models evaluated.

Many models of diameter distributions have been described (Bailey and Dell 1973; and references therein), but none seem to be available for basal area, biomass, and surface area. A particularly promising diameter distribution is the Weibull (Bailey and Dell 1973; Bailey 1974). Some models describing the distribution of canopy over live-crown length have been described for coniferous stands. Stephens (1969) studied the fit of the normal distribution to foliage weight data. He calculated the foliage distribution in the stand canopy by assuming that the distribution of foliage in each diameter class was normally distributed with mean of half and standard deviation of one-fifth of crown length in that class. His results show that the normal distribution gives a reasonable approximation to foliage weight frequency data of 10 red pine stands 20–50

TABLE 1. Summary of physical dimensions for white pine and loblolly pine sample trees used in the analysis of foliage distribution

Species	n	Basal area (cm²) Mean	Basal area (cm²) Range	Total height (m) Mean	Total height (m) Range	Foliage weight (kg) Mean	Foliage weight (kg) Range	Foliage area (m²) Mean	Foliage area (m²) Range
White pine (1968)	20	92	5–263	6.6	3.5–10.1	3.5	0.1– 8.7	69	4–179
White pine (1972)	13	150	6–401	9.8	5.0–13.8	3.0	0.1–10.2	65	1–229
Loblolly pine (1970)	5	244	117–457	15.1	14.1–16.3	3.07	0.9– 8.3	NA	

years in age. Kinerson and Fritschen (1971) assumed that the distribution of canopy surface area in each diameter class was triangular and calculated the distribution of the canopy for a naturally regenerated Douglas-fir (*Pseudotsuga menziesii* (Mirb.) Franco) stand. They showed vertical crown canopy distributions exhibiting negative, zero, or positive skew, but gave no explicit model of the distribution. Kinerson *et al.* (1974) fitted a nonlinear least squares model to cumulative crown class frequency data. The model gave a satisfactory fit to their data.

Data Source

Data for model development were derived from two stands in North Carolina where the forest ecosystems are being intensively studied. One stand is a white pine plantation near the Coweeta Hydrologic Laboratory and the other is a loblolly pine plantation at Research Triangle Park.

The white pines, covering 16.1 ha, were planted in 1957 after the indigenous hardwood forest was cut. In 1967, 1969, 1971, and 1972 diameters at breast height (D) (1.37 m) were measured on each tree on 20 randomly located 0.08-ha plots. Between 1967 and 1972, basal area increased from 7.3 to 23.4 m²/ha. The number of trees decreased from 1790 to 1760/ha.

Seven diameter classes were designated, and 20 trees were selected by stratified random sampling from the classes in the watershed population in February 1968. Sample trees were felled, and detailed measurements were made of D, total height, height of each node above the ground, length of live crown, stem diameter at 1-m intervals, and diameter of each branch. All branches were cut and transported to the laboratory where all foliage was separated from the branches by node, dried at 70 °C, and weighed. Surface area of foliage and branches was estimated from subsamples. Sampling methods and statistical analyses are described in detail by Swank and Schreuder (1974). In February 1972, 13 sample trees were collected and the same measurements were repeated.

Two neighboring loblolly pine plantations on the Duke Forest were selected for study. One was established in 1953 and the other in 1955. The 1953 plantation was inventoried completely in 1964, 1966, and 1968. In 1968, 16 trees were selected by stratified random sampling from seven diameter classes. Sample trees were cut, and their foliage, branches, stem biomass, diameters at breast height, and total heights were measured. The study area in 1964 and 1966 was a 0.16-ha plot. A small adjacent area was included in 1968 which accounts for an increase in number of trees between 1964 and 1968. In 1968, there were 2243 trees/ha and mean basal area was 49 m²/ha.

Five loblolly pine trees were randomly selected from the 1955 plantation in 1970. Thus, both plantations were measured after 16 growing seasons. For the five trees, foliage plus branch biomass by node, height to each node, and total height and diameter breast height were measured. Sampling methods and statistical analyses are described in detail by Wells *et al.* (1974). Table 1 summarizes average physical dimensions for each set of sample trees.

Data Analysis and Results

Distributions of Tree Basal Area, Biomass, and Surface Area in the Two Plantations

Each of the three years (1964, 1966, and 1968) of loblolly diameter data were tallied into seven diameter classes; the four years (1967, 1969, 1971, and 1972) of white pine

TABLE 2. A comparison by log likelihoods (ln L) of four diameter distribution models for 4 years of white pine and three years of loblolly pine data

Species and year	Number of diameter classes	Sample size	ln L Normal	Lognormal	Gamma	Weibull
White pine						
1967	7	2881	−4571	−4875	−4662	−4523
1969	10	2907	−5221	−5625	−5373	−5212
1971	12	2899	−5826	−6287	−6015	−5830
1972	12	2868	−5972	−6400	−6140	−5964
Loblolly pine						
1964	7	304	−479	−513	−484	−470
1966	7	300	−498	−512	−510	−497
1968	7	360	−594	−613	−600	−589

data were tallied into, respectively, 7, 10, 12, and 12 diameter classes. Four statistical distributions were fitted to each of the seven data sets using maximum likelihood estimates of the parameters of the distributions. Distributions were evaluated with the likelihood criteria (Cox 1961). Since basal area is equal to a constant times diameter squared, the basal area distribution can be derived from the diameter distribution. We found in an earlier study (Schreuder and Swank 1974) that, for each year considered, stem, branch, and leaf biomass and surface area are strongly related linearly to basal area. That is, $Y = a + b\,BA$, $R^2 \geqslant 0.934$ in all weighted regressions where Y represents the respective biomass and surface area variables, BA denotes basal area, and a and b are the usual regression coefficients. Therefore, we can also derive the distribution of the biomass and surface area variables.

The statistical distributions used were the normal

$$F(D) = \frac{1}{\sigma_1\sqrt{2\pi}} \int_{-\infty}^{D} e^{-(1/2)(D-\mu)^2/\sigma_2^1}\,dx \qquad -\infty < x < \infty,$$

the lognormal

$$F(D) = \frac{1}{x\sigma_2\sqrt{2\pi}} \int_{0}^{D} e^{-(1/2)(\ln x-\xi)^2/\sigma_2{}^2}dx \qquad x > 0,$$

the gamma

$$F(D) = \int_0^D \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha\Gamma(\alpha)}dx \qquad x > 0 \quad \alpha, \beta > 0,$$

where

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\,dt,$$

and the Weibull

$$F(D) = 1 - e^{-(D/\alpha)^c} \quad x > 0.$$

These distributions were selected because with them no transformations are needed for even-aged plantations. For such plantations, diameter distributions generally are considered to be symmetrical or positively skewed, and the possible range of the variable in each of these distributions equals or exceeds the range of the variable encountered in practice. Maximum likelihood estimators were calculated for all distributions, and the observed and expected fit were compared using the likelihood criteria (Table 2). The larger the value of the likelihood; the better the fit. In six of the seven cases, the Weibull distribution gave the best fit; it was second best in the other case. The normal distribution gave results almost as good as the Weibull and was best in one case. The gamma gave consistently poorer fits, and the lognormal always was worst.

We selected the Weibull distribution because it generally gave the best results. Its shape is flexible and it is easy to use once the parameters have been estimated (Johnson and Kotz 1970; Bailey and Dell 1973). In addition, it can be readily proved that if D (in cm) is distributed as Weibull with parameters α, c then basal area in square centimeters is distributed as Weibull with parameters α_1 and c_1, where $\alpha_1 = \pi\alpha^2/4$ and $c_1 = c/2$.[1] Therefore, the parameters for the basal area distributions

[1]In fact if D has the lognormal or the Weibull distribution, then $y = a_1 + a_2D^{a_3}$ also follows the lognormal or the Weibull distribution according to a personal communication from Robert L. Bailey.

can be calculated from those of the diameter distribution (Table 3).

If basal area (BA) is distributed as Weibull with parameters α_1 and c_1, then $Y = a + b\ BA$ is distributed as a 3-parameter Weibull with parameters α_2, c_1, and a or $Y - a$ is distributed as Weibull with parameters α_2 and c_1, where $\alpha_2 = b\alpha_1$. The relations of biomass and surface area to basal area have been described previously (Schreuder and Swank 1974). The biomass and surface area Weibull parameter estimates are given in Table 4.

Crown Profiles

Four steps were taken in constructing a foliar distribution of the white pine stand: (1) node height above ground level was plotted for each sample tree; (2) the total length of live crown (l) was found by taking the distance between the lowermost live node and the uppermost node for all sample trees (l is an underestimate of the total length of live crown for the stand); (3) the crown length (l) was divided into k equal-size strata, and the strata boundaries were delineated on the plot of node height; (4) the proportion of foliage area or biomass falling into each stratum was estimated as follows:

$$\hat{F}_i(i = 1, \ldots, k) = \sum_{i=1}^{7} \hat{N}_h \bar{F}_{hi}$$

where

$$\bar{F}_{hi} = \sum_{j=1}^{n_h} \hat{F}_{hij}/n_h$$

where: $\hat{F}_{hij}$ = foliage in crown height stratum i for tree j in d.b.h. (diameter breast height) stratum h;

n_h = number of sample trees in d.b.h. stratum h;

$\bar{F}_{hi}$ = average foliar area or biomass per tree in crown height stratum i from d.b.h. stratum h; and

$\hat{N}_h$ estimated total number of trees in d.b.h. stratum h.

TABLE 3. Diameter (in cm) and basal area (in cm^2) Weibull parameter estimates for 4 years of white pine and 3 years of loblolly pine data

	Diameter parameter estimates		Basal area parameter estimates	
Species and year	α	c	α_1	c_1
White pine				
1967	7.39	2.33	42.86	1.17
1969	10.17	2.64	81.16	1.32
1971	12.73	2.68	127.27	1.34
1972	13.51	2.60	143.34	1.30
Loblolly pine				
1964	15.48	5.73	188.02	2.86
1966	17.49	5.73	240.01	2.87
1968	17.46	5.84	239.17	2.92

An underlying assumption in this procedure, and in the methods of Stephens (1969) and Kinerson and Fritschen (1971), is that foliage in each whorl is equally distributed along its internode.

For the 20 white pine sample trees, crown length (l) was 9.85, the number of strata was set at eight, and the strata sizes were 1.23 m. Eight strata were also used for the second set of sample trees but l increased to 12.96, which gave a stratum size of 1.62 m. The five loblolly pine trees had an l of 8 and eight strata with an interval of 1 m. The foliage area and biomass are plotted at the midpoint of each stratum (Figure 1).

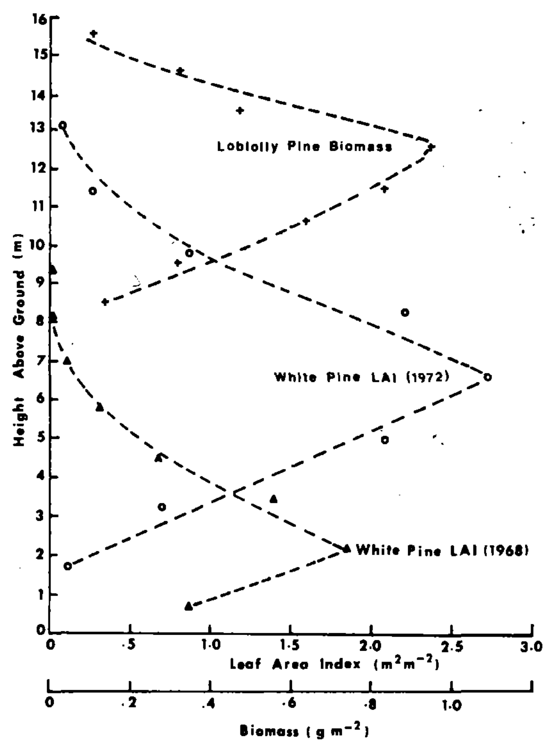

FIG. 1. Vertical distribution of foliage area for a white pine plantation at ages 10 and 14 and foliage biomass for a loblolly pine plantation at age 16.

TABLE 4. Biomass and surface area Weibull parameter estimates for the white pine plantation in 1968, 1970, and 1972 and biomass parameter estimates for the loblolly pine plantation in 1968

Character and unit of measure	α_1	c_1	a
1968 white pine			
Foliage area (m^2)	30.86	1.17	0.69
Foliage weight (g)	1505.24	1.17	−5.41
Branch area (m^2)	4.71	1.17	0.27
Branch weight (g)	5462.94	1.17	−933.21
Stem area (dm^2)	50.57	1.17	22.72
Stem weight (kg)	4.71	1.17	−0.18
1970 white pine			
Foliage area (m^2)	58.44	1.32	0.69
Foliage weight (g)	2850.34	1.32	−5.41
Branch area (m^2)	8.93	1.32	0.27
Branch weight (g)	10 344.65	1.32	−933.21
Stem area (dm^2)	123.36	1.32	25.40
Stem weight (kg)	14.61	1.32	−0.55
1972 white pine			
Foliage area (m^2)	64.50	1.30	−3.87
Foliage weight (g)	3028.77	1.30	−165.45
Branch area (m^2)	12.90	1.30	0.84
Branch weight (g)	14 523.21	1.30	−529.95
Stem area (dm^2)	217.88	1.30	25.40
Stem weight (kg)	25.80	1.30	−0.55
1968 loblolly pine			
Foliage weight (kg)	5.00	2.92	−1.00
Branch weight (kg)	10.40	2.92	−2.98
Stem weight (kg)	66.23	2.92	−4.62

TABLE 5. A comparison by log likelihoods ($\ln L$) of three crown profile biomass or surface area distribution models for 1968, 1972 white pine and 1971 loblolly pine crown profile data and maximum likelihood estimates for the Weibull distribution parameters

Species	Year	Number of crown height classes	Number of felled trees	$\ln L$ Lognormal	$\ln L$ Gamma	$\ln L$ Weibull	Weibull parameter estimates
White pine							
Surface area	1968	8	20	−1.85	−1.78	−0.76	$\hat{c} = 1.94, \hat{\alpha} = 3.11$
Biomass	1972	8	13	−2.20	−2.16	−1.16	$\hat{c} = 3.60, \hat{\alpha} = 7.64$
Surface area	1972	8	13	−2.19	−2.15	−1.14	$\hat{c} = 3.59, \hat{\alpha} = 7.50$
Loblolly pine							
Biomass	1971	8	5	−1.92	−1.92	−0.94	$\hat{c} = 7.95, \hat{\alpha} = 12.59$

In mathematically describing the crown profile data, we decided to treat the data as probabilistic samples. We divided total biomass (or surface area) in each stratum by the total biomass (or surface area) in the stand to get proportions. Treating the proportions as probabilities simplified our nonlinear estimation problem because we could apply maximum likelihood formulations to the parameter estimation. In addition, we had an objective measure to compare the models in the likelihood index.

The statistical distributions considered were the lognormal, the gamma, and the Weibull since positive skew was anticipated for some of the crown profiles. Results with the Weibull distribution were best in all four cases (Table 5).

Discussion

The consistent superiority of the Weibull distribution is remarkable. It can readily be explained in the case of the loblolly pine crown profile data since only the Weibull distribution can accommodate the negative skew there. The reason for consistent superiority in the other situations is not obvious.

A considerable advantage of the Weibull is that it can be used to describe the distributional patterns of all variables considered in a theoretically consistent manner. The form of the distribution function, once the parameters have been estimated, is simple. None of the other distribution function forms are simple, and only the lognormal can describe the distributional patterns of all variables in as consistent a manner. With the two other distributions at least two statistical distributions would be required for theoretical consistency, one of which in each case would be quite complex.

The Weibull distribution lends itself to some useful characterizations of stands. For example, white pine foliage distribution in 1968 had positive skew because of only partial crown closure, but skewness was about zero when crown closure was complete in 1972. Skewness is reflected in the parameter c for the Weibull, which goes from 1.9 in 1968, indicating positive skew, to 3.6 in 1972, indicating zero skew and close approximation to the normal distribution. The $c = 7.95$ for the older loblolly pine stand indicates that skew may be negative in older stands and that the normality assumption of Stephens (1969) may not be reasonable for some older pine stands.

We had expected that as plantations aged their diameter distribution would move towards normal. There is some evidence of this in the white pine stand (c increases some from 1967 to 1971 but decreases slightly from 1971 to 1972) but not as much as we had expected. The failure to achieve normality can be attributed to an ability of smallest trees to survive even though they are essentially not growing. The loblolly pine plantation exhibits the same negative skew from 1964 through 1968. This negative skew may be typical of fully closed stands, but it has not been reported earlier in the literature.

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